

Metrical lower bounds on the discrepancy of digital Kronecker-sequences

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Abstract

Digital Kronecker-sequences are a non-archimedean analog of classical Kronecker-sequences whose construction is based on Laurent series over a finite field. In this paper it is shown that for almost all digital Kronecker-sequences the star discrepancy satisfies $D_N^* \geq c(q, s)(\log N)^s \log \log N$ for infinitely many $N \in \mathbb{N}$, where $c(q, s) > 0$ only depends on the dimension s and on the order q of the underlying finite field, but not on N . This result shows that a corresponding metrical upper bound due to Larcher is up to some $\log \log N$ term best possible.

1 Introduction and statement of the result

For an s tuple $\alpha = (\alpha_1, \dots, \alpha_s)$ of reals the classical *Kronecker-sequence* $\mathcal{S}(\alpha) = (\mathbf{x}_n)_{n \geq 0}$ is defined by

$$\mathbf{x}_n := (\{n\alpha_1\}, \dots, \{n\alpha_s\}) \quad \text{for } n \in \mathbb{N}_0,$$

where $\{x\}$ denotes the fractional part of a real number x . It was shown by Weyl [15] that that $\mathcal{S}(\alpha)$ is uniformly distributed in the s -dimensional unit-cube $[0, 1]^s$ if and only if $1, \alpha_1, \dots, \alpha_s$ are linearly independent over \mathbb{Q} . Quantitative versions of this result can be stated in terms of star discrepancy which is defined as follows:

Let $\mathcal{S} = (\mathbf{y}_n)_{n \geq 0}$ be an infinite sequence in the s -dimensional unit-cube $[0, 1]^s$. For $\mathbf{x} = (x_1, \dots, x_s) \in [0, 1]^s$ and $N \in \mathbb{N}$ (by \mathbb{N} we denote the set of positive integers and we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) the *local discrepancy* $D(\mathbf{x}, N)$ of \mathcal{S} is the difference between the number of indices $n = 0, \dots, N-1$ for which \mathbf{y}_n belongs to the interval $[\mathbf{0}, \mathbf{x}) = \prod_{j=1}^s [0, x_j)$ and the expected number $Nx_1 \cdots x_s$ of points in $[\mathbf{0}, \mathbf{x})$ if we assume a perfect uniform distribution on $[0, 1]^s$, i.e.,

$$D(\mathbf{x}, N) = \#\{0 \leq n < N : \mathbf{x}_n \in [\mathbf{0}, \mathbf{x})\} - Nx_1 \cdots x_s.$$

Definition 1 (star discrepancy). The *star discrepancy* D_N^* of a sequence \mathcal{S} is the L_∞ -norm of the local discrepancy, i.e.,

$$D_N^*(\mathcal{S}) = \|D(\mathbf{x}, N)\|_\infty.$$

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Note that often a normalized version is used for defining the star discrepancy. A sequence \mathcal{S} is called *uniformly distributed* if and only if the normalized star discrepancy $D_N^*(\mathcal{S})/N$ tends to 0 for growing N . Furthermore, the (normalized) star discrepancy can be used to bound the integration error of a quasi-Monte Carlo algorithm based on \mathcal{S} via the well known Koksma-Hlawka inequality. For more information on uniform distribution, discrepancy and quasi-Monte Carlo integration we refer to [3, 5, 11].

Apart from the one-dimensional case $s = 1$, it is very difficult to give good estimates for the star discrepancy of concrete Kronecker-sequences. In a remarkable paper Beck [1] showed the following metrical result:

For arbitrary increasing function $\varphi(n)$ of $n \in \mathbb{N}$ we have

$$D_N^*(\mathcal{S}(\boldsymbol{\alpha})) \ll_s (\log N)^s \varphi(\log \log N) \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} \frac{1}{\varphi(n)} < \infty$$

*for almost all $\boldsymbol{\alpha} \in \mathbb{R}^s$.*¹

In particular, for almost every $\boldsymbol{\alpha} \in \mathbb{R}^s$ we have

$$D_N^*(\mathcal{S}(\boldsymbol{\alpha})) \ll_{s,\varepsilon} (\log N)^s (\log \log N)^{1+\varepsilon}$$

for every $\varepsilon > 0$, and for almost every $\boldsymbol{\alpha} \in \mathbb{R}^s$ there are infinitely many $N \in \mathbb{N}$ such that

$$D_N^*(\mathcal{S}(\boldsymbol{\alpha})) \geq c(s)(\log N)^s \log \log N$$

with a $c(s) > 0$ not depending on N .

In connection with the construction of digital sequences a “non-archimedean analog” to classical Kronecker-sequences has been introduced by Niederreiter [11, Section 4] and further investigated by Larcher and Niederreiter [8].

Let q be a prime number and let \mathbb{Z}_q be the finite field of order q . We identify \mathbb{Z}_q with the set $\{0, 1, \dots, q-1\}$ equipped with arithmetic operations modulo q . Let $\mathbb{Z}_q[x]$ be the set of all polynomials over \mathbb{Z}_q and let $\mathbb{Z}_q((x^{-1}))$ be the field of formal Laurent series

$$g = \sum_{k=w}^{\infty} a_k x^{-k} \quad \text{with } a_k \in \mathbb{Z}_q \text{ and } w \in \mathbb{Z} \text{ with } a_w \neq 0.$$

The discrete exponential evaluation ν of g is defined by $\nu(g) := -w$ ($\nu(0) := -\infty$). Furthermore, we define the “fractional part” of g by

$$\{g\} := \sum_{k=\max(1,w)}^{\infty} a_k x^{-k}.$$

Throughout the paper we associate a nonnegative integer n with q -adic expansion $n = n_0 + n_1 q + \dots + n_r q^r$ with the polynomial $n(x) = n_0 + n_1 x + \dots + n_r x^r$ in $\mathbb{Z}_q[x]$ and vice versa.

¹Here $A(N, s) \ll_s B(N, s)$ means that there exists a quantity $c(s) > 0$ which depends only on s (and not on N) such that $A(N, s) \leq c(s)B(N, s)$.

For every s -tuple $\mathbf{f} = (f_1, \dots, f_s)$ of elements of $\mathbb{Z}_q((x^{-1}))$ we define the sequence $\mathcal{S}(\mathbf{f}) = (\mathbf{x}_n)_{n \geq 0}$ by

$$\mathbf{x}_n = (\{n(x)f_1(x)\}_{|x=q}, \dots, \{n(x)f_s(x)\}_{|x=q}) \quad \text{for } n \in \mathbb{N}_0.$$

This sequence can be viewed as analog to the classical Kronecker-sequence and is therefore sometimes called a *digital Kronecker-sequence* (this terminology will be clearer in a moment).

In analogy to classical Kronecker-sequences it has been shown in [8] that a digital Kronecker-sequence $\mathcal{S}(\mathbf{f})$ is uniformly distributed in $[0, 1)^s$ if and only if $1, f_1, \dots, f_s$ are linearly independent over $\mathbb{Z}_q[x]$. The special case that the f_i are rational functions was studied in [6] and in [4].

In the analysis of digital Kronecker-sequences one can obviously restrict to the set $\overline{\mathbb{Z}}_q((x^{-1}))$ of Laurent series over \mathbb{Z}_q with $w \geq 1$, i.e. with $g = \{g\}$.

In analogy to the results of Beck here we are interested in metrical results for the star discrepancy of digital Kronecker-sequences. To tackle this problem we need to introduce a suitable probability measure on $(\overline{\mathbb{Z}}_q((x^{-1})))^s$.

By μ we denote the normalized Haar-measure on $\overline{\mathbb{Z}}_q((x^{-1}))$ and by μ_s the s -fold product measure on $(\overline{\mathbb{Z}}_q((x^{-1})))^s$. We remark that μ has the following rather simple shape: If we identify the elements $\sum_{k=1}^{\infty} t_k x^{-k}$ of $\overline{\mathbb{Z}}_q((x^{-1}))$ where $t_k \neq q-1$ for infinitely many k in the natural way with the real numbers $\sum_{k=1}^{\infty} t_k q^{-k} \in [0, 1)$, then, by neglecting the countable many elements where $t_k \neq q-1$ only for finitely many k , μ corresponds to the Lebesgue measure λ on $[0, 1)$. For example, the “cylinder set” $C(c_1, \dots, c_m)$ consisting of all elements $g = \sum_{k=1}^{\infty} a_k x^{-k}$ from $\overline{\mathbb{Z}}_q((x^{-1}))$ with $a_k = c_k$ for $k = 1, \dots, m$ and arbitrary $a_k \in \mathbb{Z}_q$ for $k \geq m+1$ has measure $\mu(C(c_1, \dots, c_m)) = q^{-m}$.

In [7] Larcher proved the following metrical upper bound on the star discrepancy of digital Kronecker-sequences.

Theorem 1 (Larcher, 1995). *Let $s \in \mathbb{N}$, let q be a prime number and let $\varepsilon > 0$. For μ_s -almost all $\mathbf{f} \in (\overline{\mathbb{Z}}_q((x^{-1})))^s$ the digital Kronecker-sequence $\mathcal{S}(\mathbf{f})$ has star discrepancy satisfying*

$$D_N^*(\mathcal{S}(\mathbf{f})) \leq c(q, s, \varepsilon)(\log N)^s (\log \log N)^{2+\varepsilon}$$

with a $c(q, s, \varepsilon) > 0$ not depending on N .

Recall that it follows from a result of Roth [12] that there exists a quantity $c(s) > 0$ such that for every sequence \mathcal{S} in $[0, 1)^s$ we have

$$D_N^*(\mathcal{S}(\mathbf{f})) \geq c(s)(\log N)^{s/2} \quad \text{for infinitely many } N \in \mathbb{N}. \quad (1)$$

For a proof, see, for example, [5, Chapter 2, Theorem 2.2]. Many people believe that the exponent $s/2$ of the logarithm in (1) can be replaced by s but until now there is no proof of this conjecture for $s \geq 2$. For $s = 1$ we have

$$D_N^*(\mathcal{S}) \geq c \log N \quad \text{for infinitely many } N \in \mathbb{N} \quad (2)$$

with a constant $c > 0$ which is independent of N . This has been shown by Schmidt [13].

It is the object of this paper to show that the metrical upper bound from Theorem 1 is best possible in the order of magnitude in N (up to some $\log \log N$ term). We will prove:

Theorem 2. *Let $s \in \mathbb{N}$ and let q be a prime number. For μ_s -almost all $\mathbf{f} \in (\overline{\mathbb{Z}}_q((x^{-1})))^s$ the digital Kronecker-sequence $\mathcal{S}(\mathbf{f})$ has star discrepancy satisfying*

$$D_N^*(\mathcal{S}(\mathbf{f})) \geq c(q, s)(\log N)^s \log \log N \quad \text{for infinitely many } N \in \mathbb{N}$$

with some $c(q, s) > 0$ not depending on N .

For the proof of Theorem 2 we use an approach similar to the technique used by Beck [1] to give a metric lower bound for the discrepancy of Kronecker sequences. In the following section we will collect some auxiliary results. The proof of Theorem 2 is then presented in Section 3.

2 Auxiliary results

For given $\mathbf{f} = (f_1, \dots, f_s) \in (\overline{\mathbb{Z}}_q((x^{-1})))^s$ with $f_j = \frac{f_{j,1}}{x} + \frac{f_{j,2}}{x^2} + \frac{f_{j,3}}{x^3} + \dots \in \overline{\mathbb{Z}}_q((x^{-1}))$ we define $\mathbb{N} \times \mathbb{N}$ matrices C_1, \dots, C_s over \mathbb{Z}_q by

$$C_j = \begin{pmatrix} f_{j,1} & f_{j,2} & f_{j,3} & \dots \\ f_{j,2} & f_{j,3} & f_{j,4} & \dots \\ f_{j,3} & f_{j,4} & f_{j,5} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

Then the elements \mathbf{x}_n of the digital Kronecker-sequence can be constructed with the following digital method: For $n \in \mathbb{N}_0$ with q -adic expansion $n = n_0 + n_1q + n_2q^2 + \dots$ (this expansion is obviously finite) we set

$$\vec{n} = (n_0, n_1, n_2, \dots)^\top \in (\mathbb{Z}_q^\mathbb{N})^\top$$

and then we put

$$\vec{x}_{n,j} := C_j \vec{n} \quad \text{for } j = 1, \dots, s$$

where all arithmetic operations are taken modulo q . Write $\vec{x}_{n,j}$ as $\vec{x}_{n,j} = (x_{n,j,1}, x_{n,j,2}, \dots)^\top$. Then the n th point \mathbf{x}_n of the sequence $\mathcal{S}(\mathbf{f})$ is given by $\mathbf{x}_n = (x_{n,1}, \dots, x_{n,s})$ where

$$x_{n,j} = x_{n,j,1}q^{-1} + x_{n,j,2}q^{-2} + \dots.$$

It follows that digital Kronecker-sequences are just special examples of digital sequences as introduced by Niederreiter in [10], see also [3, 11]. This way of describing the sequence $\mathcal{S}(\mathbf{f})$ is the reason why it is called a digital Kronecker-sequence.

We continue with some notational issues. As already mentioned we sometimes consider $j \in \mathbb{N}_0$ as elements in $\mathbb{Z}_q[x]$ and vice versa. Similarly, $f \in \overline{\mathbb{Z}}_q((x^{-1}))$ is sometimes considered as element in $[0, 1)$ and vice versa, just by substituting q for x . It should always be clear from the context what is meant. However, multiplication and addition of polynomials and Laurent series are always performed in $\mathbb{Z}_q((x^{-1}))$.

An important tool in our analysis are q -adic Walsh functions which we introduce now:

Definition 2 (q -adic Walsh functions). Let q be a prime number and let $\omega_q := \exp(2\pi i/q)$ be the q th root of unity. For $j \in \mathbb{N}_0$ with q -adic expansion $j = j_0 + j_1q + j_2q^2 + \dots$ (this expansion is obviously finite) the j th q -adic *Walsh function* ${}_q\text{wal}_j : \mathbb{R} \rightarrow \mathbb{C}$, periodic with period one, is defined as

$${}_q\text{wal}_j(x) = \omega_q^{j_0\xi_1 + j_1\xi_2 + j_2\xi_3 + \dots}$$

whenever $x \in [0, 1)$ has q -adic expansion of the form $x = \xi_1q^{-1} + \xi_2q^{-2} + \xi_3q^{-3} + \dots$ (unique in the sense that infinitely many of the digits ξ_i must be different from $q - 1$).

We collect some properties of Walsh functions. More informations can be found in [3, Appendix A].

Lemma 1. For $j, k, l \in \mathbb{Z}_q[x]$ and $f, f_1, f_2 \in \mathbb{Z}_q((x^{-1}))$ we have

1. ${}_q\text{wal}_j(kf_1 + lf_2) = {}_q\text{wal}_{jk}(f_1) {}_q\text{wal}_{jl}(f_2)$, where $kf_1 + lf_2$ is evaluated in $\mathbb{Z}_q((x^{-1}))$ and jk and jl , respectively, in $\mathbb{Z}_q[x]$;

2. ${}_q\text{wal}_j(f) {}_q\text{wal}_k(f) = {}_q\text{wal}_{j+k}(f)$;

3.

$$\sum_{x=0}^{q^m-1} {}_q\text{wal}_k(x/q^m) \overline{{}_q\text{wal}_l(x/q^m)} = \begin{cases} 1 & \text{if } x^m | k - l, \\ 0 & \text{otherwise,} \end{cases}$$

(orthonormality of Walsh functions); and

4. $\int_0^1 {}_q\text{wal}_k(x) dx = 0$ whenever $k \neq 0$.

Proof. These are standard properties of Walsh functions and are easily deduced from their definition. Alternatively we refer to [3, Appendix A]. \square

We need some notation. For $m \in \mathbb{N}_0$ let

$$\begin{aligned} \mathbb{Q}(q^m) &= \{x = rq^{-m} \in [0, 1) : r = 0, \dots, q^m - 1\}, \\ \mathbb{Q}^s(q^m) &= \{\mathbf{x} = (x_1, \dots, x_s) \in [0, 1)^s : x_j \in \mathbb{Q}(q^m) \text{ for } j = 1, \dots, s\}. \end{aligned}$$

Lemma 2. Let $\mathcal{S}(\mathbf{f}) = (\mathbf{x}_n)_{n \geq 0}$ be a digital Kronecker-sequence generated by an s -tuple $\mathbf{f} = (f_1, \dots, f_s) \in (\mathbb{Z}_q((x^{-1})))^s$. Let $N \in \mathbb{N}$ with base q expansion $N = N_{m-1}q^{m-1} + \dots + N_1q + N_0$ and let $\mathbf{x} = (x_1, \dots, x_s) \in \mathbb{Q}^s(q^m)$. Then we have

$$D(\mathbf{x}, N) = \sum_{\substack{k_1, \dots, k_s=0 \\ (k_1, \dots, k_s) \neq (0, \dots, 0)}}^{q^m-1} \left(\prod_{j=1}^s J_{k_j}(x_j) \right) G(N, w(k_1, \dots, k_s)),$$

where for $k = \kappa q^{a-1} + k'$ with $a \in \mathbb{N}$, $1 \leq \kappa < q$ and $0 \leq k' < q^{a-1}$ we have

$$J_k(x) = \frac{1}{q^a} \left(\frac{1}{1 - \omega_q^{-\kappa}} \overline{{}_q\text{wal}_{k'}(x)} + \left(\frac{1}{2} + \frac{1}{\omega_q^{-\kappa} - 1} \right) \overline{{}_q\text{wal}_k(x)} \right) \quad (3)$$

$$+ \sum_{c=1}^{m-a} \sum_{l=1}^{q-1} \frac{1}{q^c(\omega_q^l - 1)} \overline{{}_q\text{wal}_{lq^a+c-1+k}(x)} - \frac{1}{2q^{m-a}} \overline{{}_q\text{wal}_k(x)} \quad (4)$$

and for $k = 0$ we have

$$J_0(x) = \frac{1}{2} + \sum_{c=1}^m \sum_{l=1}^{q-1} \frac{1}{q^c(\omega_q^l - 1)} \frac{1}{{}_q\text{wal}_{lq^{c-1}}(x)} - \frac{1}{2q^m},$$

and where

$$G(N, w(k_1, \dots, k_s)) = \left[\omega_q^{b_{w+1}N_{w+1} + \dots + b_{m-1}N_{m-1}} q^w \left(\frac{\omega_q^{b_w N_w} - 1}{\omega_q^{b_w} - 1} + \omega_q^{b_w N_w} \left\{ \frac{N}{q^w} \right\} \right) \right]$$

with

$$w = w(k_1, \dots, k_s) = -\nu \left(\left\{ \sum_{j=1}^s k_j f_j \right\} \right).$$

However, if $w \geq m$, then we put $G(N, w) = N$.

Proof. This follows directly from [9, Lemma 7] and the construction of $\mathcal{S}(\mathbf{f})$ in terms of matrices C_1, \dots, C_s . \square

Lemma 3. For $m \in \mathbb{N}$ and $k_1, \dots, k_s \in \mathbb{Z}_q[x]$, not all of them 0, let

$$M_m(k_1, \dots, k_s) := \{(f_1, \dots, f_s) \in (\overline{\mathbb{Z}}_q((x^{-1}))^s : \nu(\{k_1 f_1 + \dots + k_s f_s\}) \leq -m\}.$$

Then we have

$$\mu_s(M_m(k_1, \dots, k_s)) = \frac{1}{q^{m-1}}.$$

Proof. Let χ be the characteristic function of the interval $[0, q^{-(m-1)})$. Then χ has a finite Walsh series representation in base q of the form

$$\chi(x) = \sum_{i=0}^{q^{m-1}-1} a_i {}_q\text{wal}_i(x)$$

with $a_0 = q^{-(m-1)}$, see [3, Lemma 3.9]. Now

$$\begin{aligned} \mu_s(M_m(k_1, \dots, k_s)) &= \int_{[0,1]^s} \chi(\{k_1 f_1 + \dots + k_s f_s\}) \, df_1 \dots df_s \\ &= a_0 + \sum_{i=1}^{q^{m-1}-1} a_i \int_{[0,1]^s} {}_q\text{wal}_i(k_1 f_1 + \dots + k_s f_s) \, df_1 \dots df_s \\ &= \frac{1}{q^{m-1}} + \sum_{i=1}^{q^{m-1}-1} a_i \prod_{j=1}^s \int_0^1 {}_q\text{wal}_{ik_j}(f_j) \, df_j \\ &= \frac{1}{q^{m-1}}, \end{aligned}$$

since at least one of the k_j is different from zero and for such a k_j we have $\int_0^1 {}_q\text{wal}_{ik_j}(f_j) \, df_j = 0$ according to Lemma 1. \square

Lemma 4. Let $\overline{P} \subseteq (\mathbb{Z}_q[x] \setminus \{0\})^s$. For $(k_1, \dots, k_s) \in \overline{P}$ with $\deg(k_j) = r_j$ for $j = 1, \dots, s$ let $\beta_1(k_1), \dots, \beta_s(k_s) \in \mathbb{Z}_q[x]$ be polynomials which satisfy $\beta_j(k_j) = 0$ or $\gcd(\beta_j(k_j), k_j) = 1$ for all $j = 1, \dots, s$, but not all of them equal to zero. Let

$$\begin{aligned} \widetilde{M} &:= \{(f_1, \dots, f_s) \in (\overline{\mathbb{Z}}_q((x^{-1})))^s : \nu(\{k_1 f_1 + \dots + k_s f_s\}) \leq -(r_1 + \dots + r_s), \\ &\quad \nu(\{(k_1 + \beta_1(k_1))f_1 + \dots + (k_s + \beta_s(k_s))f_s\}) \leq -\lfloor (r_1 + \dots + r_s)/2 \rfloor \\ &\quad \text{for infinitely many } (k_1, \dots, k_s) \in \overline{P} \text{ with } \gcd(k_1, \dots, k_s) = 1\}. \end{aligned}$$

Then we have

$$\mu_s(\widetilde{M}) = 0.$$

Proof. For given k_1, \dots, k_s with $\gcd(k_1, \dots, k_s) = 1$ and not all $k_j = 1$ let

$$M_1(k_1, \dots, k_s) := \{(f_1, \dots, f_s) : \nu(\{k_1 f_1 + \dots + k_s f_s\}) \leq -(r_1 + \dots + r_s)\}$$

and

$$\begin{aligned} M_2(k_1, \dots, k_s) &:= \left\{ (f_1, \dots, f_s) : \right. \\ &\quad \left. \nu(\{(k_1 + \beta_1(k_1))f_1 + \dots + (k_s + \beta_s(k_s))f_s\}) \leq -\left\lfloor \frac{r_1 + \dots + r_s}{2} \right\rfloor \right\}. \end{aligned}$$

Let $m = r_1 + \dots + r_s$, let χ_1 be the characteristic function of the interval $[0, q^{-(m-1)})$ and let χ_2 be the characteristic function of the interval $[0, q^{\lfloor m/2 \rfloor - 1})$. Then we have finite Walsh series representation of χ_1 and χ_2 in base q given by

$$\chi_1 = \sum_{i=0}^{q^{m-1}-1} a_i^{(1)} {}_q\text{wal}_i \quad \text{and} \quad \chi_2 = \sum_{i=0}^{q^{\lfloor m/2 \rfloor - 1} - 1} a_i^{(2)} {}_q\text{wal}_i$$

with

$$a_0^{(1)} = \frac{1}{q^{m-1}} \quad \text{and} \quad a_0^{(2)} = \frac{1}{q^{\lfloor m/2 \rfloor - 1}}.$$

Now we have

$$\begin{aligned} &\mu_s(M_1(k_1, \dots, k_s) \cap M_2(k_1, \dots, k_s)) \\ &= \int_{[0,1]^s} \chi_1(\{k_1 f_1 + \dots + k_s f_s\}) \chi_2(\{(k_1 + \beta_1(k_1))f_1 + \dots + (k_s + \beta_s(k_s))f_s\}) \, df_1 \dots df_s \\ &= a_0^{(1)} a_0^{(2)} + \sum_{\substack{i,j \\ (i,j) \neq (0,0)}} a_i^{(1)} a_j^{(2)} \int_{[0,1]^s} {}_q\text{wal}_i(k_1 f_1 + \dots + k_s f_s) \\ &\quad \times {}_q\text{wal}_j((k_1 + \beta_1(k_1))f_1 + \dots + (k_s + \beta_s(k_s))f_s) \, df_1 \dots df_s. \end{aligned}$$

The integral in the last sum equals

$$\prod_{l=1}^s \int_0^1 {}_q\text{wal}_{ik_l + j(k_l + \beta_l(k_l))}(f_l) \, df_l$$

and this is zero unless we have

$$ik_l + j(k_l + \beta_l(k_l)) = 0 \quad \forall l = 1, \dots, s. \quad (5)$$

This certainly cannot hold if $i = 0$ or $j = 0$. Let $i, j \neq 0$. If $\beta_{\bar{l}}(k_{\bar{l}}) = 0$ for some \bar{l} and if (5) holds, then $ik_{\bar{l}} + jk_{\bar{l}} = 0$ and hence $i + j = 0$. Therefore we have $j\beta_l(k_l) = 0$ for all $l = 1, \dots, s$ and hence $\beta_l(k_l) = 0$ for all $l = 1, \dots, s$ what is a contradiction. This means: If (5) holds, then $i, j \neq 0$ and $\beta_l(k_l) \neq 0$ for all $l = 1, \dots, s$. Hence for all $l = 1, \dots, s$ we have $\gcd(k_l, k_l + \beta_l(k_l)) = 1$.

Now, if (5) holds, for any l, l' with $l \neq l'$ we have

$$ik_l + j(k_l + \beta_l(k_l)) = 0 \quad \text{and} \quad ik_{l'} + j(k_{l'} + \beta_{l'}(k_{l'})) = 0,$$

hence

$$\begin{aligned} 0 &= ik_l k_{l'} + j(k_{l'} + j(k_l + \beta_l(k_l)))k_{l'} \\ &= -k_l j(k_{l'} + \beta_{l'}(k_{l'})) + j(k_l + \beta_l(k_l))k_{l'} \\ &= j(\beta_l(k_l)k_{l'} - \beta_{l'}(k_{l'})k_l), \end{aligned}$$

and therefore

$$\beta_{l'}(k_{l'})k_l = \beta_l(k_l)k_{l'}.$$

Since $\gcd(k_l, \beta_l) = 1$, we conclude that $k_l | k_{l'}$ for all l' . Since $\gcd(k_1, \dots, k_s) = 1$ it follows that $k_l = 1$ and, since l was arbitrary, $(k_1, \dots, k_s) = (1, \dots, 1)$, a contradiction to the assumptions. So

$$\mu_s(M_1(k_1, \dots, k_s) \cap M_2(k_1, \dots, k_s)) = a_0^{(1)} a_0^{(2)} = \frac{1}{q^{m+[m/2]-2}}$$

and

$$\mu_s(\widetilde{M}) \leq \lim_{R \rightarrow \infty} \sum_{\substack{r_1, \dots, r_s \\ r_1 + \dots + r_s \geq R}} \sum_{\substack{k_1, \dots, k_s \\ \deg(k_i) = r_i}} \frac{1}{q^{r_1 + \dots + r_s + [(r_1 + \dots + r_s)/2] - 2}} = 0.$$

□

Lemma 5. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $(A_n)_{n \geq 1}$ be a sequence of sets $A_n \in \mathcal{A}$ such that

$$\sum_{n=1}^{\infty} \mu(A_n) = \infty.$$

Then the set A of points falling in infinitely many sets A_n is of measure

$$\mu(A) \geq \limsup_{Q \rightarrow \infty} \frac{\left(\sum_{n=1}^Q \mu(A_n) \right)^2}{\sum_{n,m=1}^Q \mu(A_n \cap A_m)}.$$

Proof. This is [14, Lemma 5 in Chapter I]. A proof can be found there. □

Lemma 6. Let $P \subseteq (\mathbb{Z}_q[x] \setminus \{0\})^s$ such that $(1, \dots, 1) \notin P$ and for each $(k_1, \dots, k_s) \in P$ we have $\gcd(k_1, \dots, k_s) = 1$. Let

$$\begin{aligned} \overline{M} = \{ (f_1, \dots, f_s) \in (\mathbb{Z}_q((x^{-1})))^s : \nu(\{k_1 f_1 + \dots + k_s f_s\}) \leq -F(r_1, \dots, r_s) \\ \text{for infinitely many } (k_1, \dots, k_s) \in P \}, \end{aligned}$$

where $r_i = \deg(k_i)$, and $F : \mathbb{N}_0^s \rightarrow \mathbb{N}$ is such that

$$\sum_{(k_1, \dots, k_s) \in P} \frac{1}{q^{F(r_1, \dots, r_s)}} = \infty.$$

Then

$$\mu_s(\overline{M}) = 1.$$

Proof. For given $(k_1, \dots, k_s) \in P$ let

$$M(k_1, \dots, k_s) := \{(f_1, \dots, f_s) \in (\mathbb{Z}_q((x^{-1})))^s : \nu(\{k_1 f_1 + \dots + k_s f_s\}) \leq -F(r_1, \dots, r_s)\}.$$

With the same proof as for Lemma 3 we have

$$\mu_s(M(k_1, \dots, k_s)) = \frac{1}{q^{F(r_1, \dots, r_s)-1}}$$

and hence

$$\sum_{(k_1, \dots, k_s) \in P} \mu_s(M(k_1, \dots, k_s)) = q \sum_{(k_1, \dots, k_s) \in P} \frac{1}{q^{F(r_1, \dots, r_s)}} = \infty,$$

and we can use Lemma 5 to obtain

$$\mu_s(\overline{M}) \geq \lim_{R \rightarrow \infty} \frac{\left(\sum_{\substack{(k_1, \dots, k_s) \in P \\ r_1 + \dots + r_s \leq R}} \mu_s(M(k_1, \dots, k_s)) \right)^2}{\sum_{\substack{(k_1, \dots, k_s) \in P \\ (l_1, \dots, l_s) \in P \\ \sum \deg(k_j) + \sum \deg(l_j) \leq R}} \mu_s(M(k_1, \dots, k_s) \cap M(l_1, \dots, l_s))}. \quad (6)$$

Proceeding in the same way as in the proof of Lemma 4 we obtain

$$\mu_s(M(k_1, \dots, k_s) \cap M(l_1, \dots, l_s)) = \mu_s(M(k_1, \dots, k_s)) \mu_s(M(l_1, \dots, l_s))$$

provided that for all $(i, j) \neq (0, 0)$ we have that

$$ik_u + jl_u = 0 \quad (7)$$

does not hold for all $u = 1, \dots, s$. Of course, by the definition of P the condition (7) can only hold if $i, j \neq 0$. If for $u \neq v$ we have

$$ik_u + jl_u = 0 \quad \text{and} \quad ik_v + jl_v = 0,$$

then

$$0 = ik_u k_v + jl_u k_v = k_u(-jl_v) + jl_u k_v = j(l_u k_v - k_u l_v).$$

Hence, if (7) holds for all u , then we have

$$l_u k_v = k_u l_v$$

for all $u, v = 1, \dots, s$ what is a contradiction since $\gcd(k_1, \dots, k_s) = \gcd(l_1, \dots, l_s) = 1$ and $(k_1, \dots, k_s) \neq (1, \dots, 1)$ and $(l_1, \dots, l_s) \neq (1, \dots, 1)$ unless $(k_1, \dots, k_s) = (l_1, \dots, l_s)$.

If we denote the summands of the sum in the denominator of (6) in any order by a_1, a_2, \dots, a_Q , then the expression on the right hand side of (6) can be written as

$$\lim_{Q \rightarrow \infty} \frac{\left(\sum_{k=1}^Q a_k\right)^2}{\left(\sum_{k=1}^Q a_k\right)^2 + \sum_{k=1}^Q a_k - \sum_{k=1}^Q a_k^2}. \quad (8)$$

Since $0 \leq a_k \leq 1$ for all k , and since $\lim_{Q \rightarrow \infty} \sum_{k=1}^Q a_k = \infty$ the limit in (8) is one and the result follows. \square

3 The proof of Theorem 2

We use the representation of $D(\mathbf{x}, N)$ given in Lemma 2. For any $\mathbf{k}^* = (k_1^*, \dots, k_s^*) \in \mathbb{N}^s$ with the property that each of the k_i^* is of the form

$$k_i^* = q^{a_i^*-1} + q^{a_i^*-2} + l_i^*$$

with some $a_i^* \geq 3$ and some $0 \leq l_i^* < q^{a_i^*-2}$ we put

$$\begin{aligned} \Lambda &:= \Lambda(\mathbf{k}^*) := \sum_{\mathbf{x} \in \mathbb{Q}^s(q^m)} D(\mathbf{x}, N) {}_q\text{wal}_{\mathbf{k}^*}(\mathbf{x}) \\ &= \sum_{\substack{k_1, \dots, k_s=0 \\ (k_1, \dots, k_s) \neq (0, \dots, 0)}}^{q^m-1} \left[\prod_{j=1}^s \sum_{x_j \in \mathbb{Q}(q^m)} J_{k_j}(x_j) {}_q\text{wal}_{k_j^*}(x_j) \right] G(N, w(k_1, \dots, k_s)). \end{aligned}$$

By the definition of the J_k and by the orthonormality of Walsh functions (see Lemma 1) we have

$$\theta(k) := \sum_{x \in \mathbb{Q}(q^m)} J_k(x) {}_q\text{wal}_{k^*}(x) = 0$$

unless we are in one of the following three cases (with $k = \kappa q^{a^*-1} + k'$ and $k^* = q^{a^*-1} + (k^*)'$):

1. k is such that $k' = k^*$, i.e., $k = \kappa q^{a^*+c-1} + k^*$ for some $c \in \mathbb{N}$ and $\kappa \in \{1, \dots, q-1\}$.

In this case we have

$$\theta(k) = \frac{1}{q^{a^*+c}} \frac{1}{1 - \omega_q^{-\kappa}}.$$

2. k is such that $k = k^*$. In this case we have

$$\theta(k) = \frac{1}{q^{a^*}} \left(\frac{1}{2} + \frac{1}{\omega_q - 1} \right) - \frac{1}{2q^m}.$$

3. k is such that $k = (k^*)' = q^{a^*-2} + l^*$. In this case we have

$$\theta(k) = \frac{1}{q^{a^*}} \frac{1}{\omega_q - 1}.$$

We write $(k_j^*)' =: \tilde{k}_j = q^{a_j^*-2} + \dots$ (note that k_j^* is uniquely determined by \tilde{k}_j),

$$\begin{aligned}\beta_j(\tilde{k}_j, 0) &:= 0 \\ \beta_j(\tilde{k}_j, 1) &:= q^{a_j^*-1}\end{aligned}$$

and for $t \in \mathbb{N}_0$ and $u_j \in \{tq - t + 2, \dots, tq - t + q\}$ we put

$$\beta_j(\tilde{k}_j, u_j) = q^{a_j^*-1} + q^{a_j^*+t}(u_j - (tq - t + 1)).$$

Then for $u_j \geq 2$ we have

$$\tilde{k}_j + \beta_j(\tilde{k}_j, u_j) = (k_j^*)' + q^{a_j^*-1} + q^{a_j^*+t}(u_j - (tq - t + 1)) = k_j^* + q^{a_j^*+t}\kappa,$$

where $\kappa = u_j - (tq - t + 1)$. Hence, according to Case 1, we have

$$|\theta(\tilde{k}_j + \beta_j(\tilde{k}_j, u_j))| \leq c_1(q) \frac{1}{q^{a_j^*+t+1}} \leq c_1(q) \frac{1}{q^{a_j^*+u_j/q}}$$

for some $c_1(q) > 0$. Similarly,

$$\tilde{k}_j + \beta_j(\tilde{k}_j, 0) = (k_j^*)'$$

and hence, according to Case 3, we have

$$|\theta(\tilde{k}_j + \beta_j(\tilde{k}_j, 0))| \leq c_2(q) \frac{1}{q^{a_j^*}}$$

for some $c_2(q) > 0$, and

$$\tilde{k}_j + \beta_j(\tilde{k}_j, 1) = (k_j^*)' + q^{a_j^*-1} = k_j^*$$

and hence, according to Case 2, we have

$$|\theta(\tilde{k}_j + \beta_j(\tilde{k}_j, 1))| \leq c_3(q) \frac{1}{q^{a_j^*}}$$

for some $c_3(q) > 0$. Summing up, for all $u_j \geq 0$ we have

$$|\theta(\tilde{k}_j + \beta_j(\tilde{k}_j, u_j))| \leq c_4(q) \frac{1}{q^{a_j^*+u_j/q}} \quad (9)$$

for some $c_4(q) > 0$.

Now we have

$$\Lambda = \sum_{u_1, \dots, u_s \geq 0} \left[\prod_{j=1}^s \theta(\tilde{k}_j + \beta_j(\tilde{k}_j, u_j)) \right] G(N, w(\tilde{k}_1 + \beta_1(\tilde{k}_1, u_1), \dots, \tilde{k}_s + \beta_s(\tilde{k}_s, u_s)))$$

where the summation is over all u_j with $\tilde{k}_j + \beta_j(\tilde{k}_j, u_j) < q^m$ for all $j = 1, \dots, s$. Then for any $J \in \mathbb{N}$ we have

$$|\Lambda| \geq \left| \left[\prod_{j=1}^s \theta(\tilde{k}_j) \right] G(N, w(\tilde{k}_1, \dots, \tilde{k}_s)) \right| \quad (10)$$

$$\begin{aligned}
& - \left| \sum_{\substack{0 \leq u_1, \dots, u_s \leq J \\ (u_1, \dots, u_s) \neq (0, \dots, 0)}} \left[\prod_{j=1}^s \theta(\tilde{k}_j + \beta_j(\tilde{k}_j, u_j)) \right] G(N, w(\tilde{k}_1 + \beta_1(\tilde{k}_1, u_1), \dots, \tilde{k}_s + \beta_s(\tilde{k}_s, u_s)) \right| \\
& - \left| \sum_{\substack{u_1, \dots, u_s \geq 0 \\ \exists j: u_j > J}} \left[\prod_{j=1}^s \theta(\tilde{k}_j + \beta_j(\tilde{k}_j, u_j)) \right] G(N, w(\tilde{k}_1 + \beta_1(\tilde{k}_1, u_1), \dots, \tilde{k}_s + \beta_s(\tilde{k}_s, u_s)) \right|.
\end{aligned}$$

Note that $|G(N, w(k_1, \dots, k_s))| \leq qN$ always. Therefore and using (9) for the last sum in (10) we have

$$|\Sigma| \leq \frac{qN}{q^{a_1^* + \dots + a_s^*}} \sum_{\substack{u_1, \dots, u_s \geq 0 \\ \exists j: u_j > J}} q^{-\frac{u_1}{q} - \dots - \frac{u_s}{q}} \leq c_5(q, s) \frac{N}{q^{a_1^* + \dots + a_s^*}} \frac{1}{q^{J/q}},$$

with some $c_5(q, s) > 0$ depending only on q and on s .

Let the function F from Lemma 6 be such that

$$q^{F(r_1, \dots, r_s)} = q^{r_1 + \dots + r_s} (r_1 + \dots + r_s)^s \log(r_1 + \dots + r_s). \quad (11)$$

Let P from Lemma 6 be given by

$$\begin{aligned}
P = & \left\{ (k_1, \dots, k_s) \in (\mathbb{Z}_q[x] \setminus \{0\})^s : (k_1, \dots, k_s) \neq (1, \dots, 1), \gcd(k_1, \dots, k_s) = 1 \right. \\
& k_i = q^{a_i-1} + \ell_i \text{ for some } a_i \in \mathbb{N} \text{ and } 0 \leq \ell_i < q^{a_i-1} \text{ for all } i = 1, \dots, s \\
& \left. \text{and } \gcd\left(k_i, x \prod_{j=1}^J \prod_{\kappa=1}^{q-1} (1 + \kappa x^j)\right) = 1 \text{ for all } i = 1, \dots, s \right\}. \quad (12)
\end{aligned}$$

Lemma 7. *With F as in (11) and P as in (12) we have*

$$\sum_{(k_1, \dots, k_s) \in P} \frac{1}{q^{F(r_1, \dots, r_s)}} = \infty,$$

where $r_i = \deg(k_i)$.

Proof. We put

$$T := \sum_{(k_1, \dots, k_s) \in P} \frac{1}{q^{F(r_1, \dots, r_s)}}.$$

Let $W_q(a)$ be the set of all monic polynomials over \mathbb{Z}_q with degree a , i.e.

$$W_q(a) = \{k \in \mathbb{Z}_q[x] : \deg(k) = a \text{ and } k \text{ is monic}\}.$$

Put $p := x \prod_{j=1}^J \prod_{\kappa=1}^{q-1} (1 + \kappa x^j) = up_1^{\alpha_1} \dots p_r^{\alpha_r}$ with $u \in \mathbb{Z}_q$, irreducible factors $p_1, \dots, p_r \in \mathbb{Z}_q[x]$ and $\alpha_1, \dots, \alpha_r \in \mathbb{N}$. Then we have

$$\begin{aligned}
T = & \sum_{\substack{a_1, \dots, a_s \\ a_1 + \dots + a_s \neq 0}} \frac{1}{q^{F(a_1, \dots, a_s)}} \underbrace{\sum_{\substack{k_1 \in W_q(a_1) \\ \gcd(k_1, p)=1}} \dots \sum_{\substack{k_s \in W_q(a_s) \\ \gcd(k_s, p)=1}} 1}_{\gcd(k_1, \dots, k_s)=1}. \quad (13)
\end{aligned}$$

Let μ_q the polynomial analog to the Möbius- μ function defined by $\mu_q(a) = 1$ and $\mu_q(af) = \mu(f)$ for $a \in \mathbb{Z}_q$ and $f \in \mathbb{Z}_q[x]$, $\mu_q(f) = 0$ if there exists an irreducible $g \in \mathbb{Z}_q[x]$ with $g^2|f$ and $\mu_q(f) = (-1)^\rho$ if f splits up in ρ different irreducible factors. We just remark that μ_q is multiplicative and refer to [2, p. 42] for more informations.

First we consider the case $s = 1$. Then the inner sum in (13) reduces to (we omit the index “1” for the sake of simplicity)

$$\begin{aligned} \sum_{\substack{k \in W_q(a) \\ \gcd(k,p)=1}} 1 &= \sum_{k \in W_q(a)} \sum_{\ell | \gcd(k,p)} \mu_q(\ell) \\ &= \sum_{\ell | p} \mu_q(\ell) \sum_{\substack{k \in W_q(a) \\ \ell | k}} 1 \\ &= \sum_{\ell | p} \mu_q(\ell) \sum_{\substack{c \in \mathbb{Z}_q[x] \\ \ell c \in W_q(a)}} 1. \end{aligned}$$

If $\ell c \in W_q(a)$, then the leading coefficient of the polynomial c is uniquely determined by ℓ and $\deg(c) = a - \deg(\ell)$. Hence we have

$$\sum_{\substack{c \in \mathbb{Z}_q[x] \\ \ell c \in W_q(a)}} 1 = q^{a - \deg(\ell)}$$

and therefore we obtain

$$\sum_{\substack{k \in W_q(a) \\ \gcd(k,p)=1}} 1 = q^a \sum_{\ell | p} \frac{\mu_q(\ell)}{q^{\deg(\ell)}}.$$

Using the factorization of p we now obtain

$$\begin{aligned} \sum_{\substack{k \in W_q(a) \\ \gcd(k,p)=1}} 1 &= q^a \sum_{d_1=0}^{\alpha_1} \dots \sum_{d_r=0}^{\alpha_r} \frac{\mu_q(p_1^{d_1} \dots p_r^{d_r})}{q^{\deg(p_1^{d_1} \dots p_r^{d_r})}} \\ &= q^a \prod_{j=1}^r \sum_{d=0}^{\alpha_j} \frac{\mu_q(p_j^d)}{q^{\deg(p_j^d)}} \\ &= q^a \prod_{j=1}^r \left(1 - \frac{1}{q^{\deg(p_j)}} \right) \\ &\geq q^a \frac{1}{2^r}. \end{aligned}$$

Inserting this result into (13) yields

$$T \geq \frac{1}{2^r} \sum_{a=1}^{\infty} \frac{1}{a \log a} = \infty$$

as claimed.

Now assume that $s \geq 2$. As above we begin by studying the inner sum in (13). We have

$$\underbrace{\sum_{\substack{k_1 \in W_q(a_1) \\ \gcd(k_1,p)=1}} \dots \sum_{\substack{k_s \in W_q(a_s) \\ \gcd(k_s,p)=1}}}_{\gcd(k_1, \dots, k_s)=1} 1 = \sum_{\substack{k_1 \in W_q(a_1) \\ \gcd(k_1,p)=1}} \dots \sum_{\substack{k_s \in W_q(a_s) \\ \gcd(k_s,p)=1}} \sum_{\ell | \gcd(k_1, \dots, k_s)} \mu_q(\ell)$$

$$= \sum_{\substack{\ell \in \mathbb{Z}_q[x] \\ \deg(\ell) \leq \min(a_1, \dots, a_s)}} \mu_q(\ell) \prod_{i=1}^s \left(\sum_{\substack{k_i \in W_q(a_i) \\ \gcd(k_i, p)=1 \\ \ell | k_i}} 1 \right).$$

For any factor of the above product we have (we omit the index “ i ” for the sake of simplicity)

$$\sum_{\substack{k \in W_q(a) \\ \gcd(k, p)=1 \\ \ell | k}} 1 = \sum_{\substack{c \in \mathbb{Z}_q[x] \\ \ell c \in W_q(a) \\ \gcd(\ell c, p)=1}} 1 = \begin{cases} 0 & \text{if } \gcd(\ell, p) > 1, \\ \sum_{\substack{c \in \mathbb{Z}_q[x] \\ \ell c \in W_q(a) \\ \gcd(c, p)=1}} 1 & \text{if } \gcd(\ell, p) = 1. \end{cases}$$

Using the same arguments as above it can be shown that

$$\sum_{\substack{c \in \mathbb{Z}_q[x] \\ \ell c \in W_q(a) \\ \gcd(c, p)=1}} 1 = q^{a - \deg(\ell)} A(p),$$

where $A(p) := \prod_{j=1}^r (1 - q^{-\deg(p_j)}) \geq 2^{-r}$. Hence we obtain

$$\begin{aligned} \underbrace{\sum_{\substack{k_1 \in W_q(a_1) \\ \gcd(k_1, p)=1}} \dots \sum_{\substack{k_s \in W_q(a_s) \\ \gcd(k_s, p)=1}} 1}_{\gcd(k_1, \dots, k_s)=1} &= \sum_{\substack{\deg(\ell) \leq \min(a_1, \dots, a_s) \\ \gcd(\ell, p)=1}} \frac{\mu_q(\ell)}{q^{s \deg(\ell)}} q^{a_1 + \dots + a_s} A(p)^s \\ &\geq q^{a_1 + \dots + a_s} A(p)^s \inf_{x \in \mathbb{N}} \sum_{\substack{\deg(\ell) \leq x \\ \gcd(\ell, p)=1}} \frac{\mu_q(\ell)}{q^{s \deg(\ell)}}. \end{aligned}$$

We show that

$$B := \inf_{x \in \mathbb{N}} \sum_{\substack{\deg(\ell) \leq x \\ \gcd(\ell, p)=1}} \frac{\mu_q(\ell)}{q^{s \deg(\ell)}} \geq \frac{q-1}{4}. \quad (14)$$

For any $x \in \mathbb{N}$ we have

$$\begin{aligned} \sum_{\substack{\deg(\ell) \leq x \\ \gcd(\ell, p)=1}} \frac{\mu_q(\ell)}{q^{s \deg(\ell)}} &= \sum_{\substack{\deg(\ell)=0 \\ \gcd(\ell, p)=1}} 1 + \sum_{\substack{1 \leq \deg(\ell) \leq x \\ \gcd(\ell, p)=1}} \frac{\mu_q(\ell)}{q^{s \deg(\ell)}} \\ &= q - 1 + \sum_{\substack{1 \leq \deg(\ell) \leq x \\ \gcd(\ell, p)=1}} \frac{\mu_q(\ell)}{q^{s \deg(\ell)}}. \end{aligned}$$

for the last sum we have

$$\begin{aligned} \left| \sum_{\substack{1 \leq \deg(\ell) \leq x \\ \gcd(\ell, p)=1}} \frac{\mu_q(\ell)}{q^{s \deg(\ell)}} \right| &\leq \frac{1}{q^s} \sum_{\substack{\deg(\ell)=1 \\ \gcd(\ell, p)=1}} 1 + \sum_{d=2}^{\infty} \frac{1}{q^{sd}} \sum_{\deg(\ell)=d} 1 \\ &\leq \frac{(q-1)^2}{q^s} + \frac{q-1}{q^{s-1}(q^{s-1}-1)}, \end{aligned}$$

where we used $\sum_{\substack{\deg(\ell)=1 \\ \gcd(\ell,p)=1}} 1 \leq (q-1)^s$ since $x|p$. Hence it follows that

$$\sum_{\substack{\deg(\ell) \leq x \\ \gcd(\ell,p)=1}} \frac{\mu_q(\ell)}{q^{s \deg(\ell)}} \geq (q-1) \left(1 - \frac{q-1}{q^s} - \frac{1}{q^{s-1}(q^{s-1}-1)} \right) \geq \frac{q-1}{4}$$

and hence (14) is shown.

Hence

$$\underbrace{\sum_{\substack{k_1 \in W_q(a_1) \\ \gcd(k_1,p)=1}} \cdots \sum_{\substack{k_s \in W_q(a_s) \\ \gcd(k_s,p)=1}}}_{\gcd(k_1, \dots, k_s)=1} 1 \geq q^{a_1 + \dots + a_s} A(p)^s \frac{q-1}{4}.$$

Inserting this in (13) we obtain

$$\begin{aligned} T &\geq A(p)^s \frac{q-1}{4} \sum_{\substack{a_1, \dots, a_s \\ a_1 + \dots + a_s \neq 0}} \frac{1}{q^{F(a_1, \dots, a_s)}} q^{a_1 + \dots + a_s} \\ &\geq \frac{1}{2^{rs}} \frac{q-1}{4} \sum_{d=1}^{\infty} \frac{1}{d^s \log d} \sum_{\substack{a_1, \dots, a_s=0 \\ a_1 + \dots + a_s = d}}^{\infty} 1 \\ &= \frac{1}{2^{rs}} \frac{q-1}{4} \sum_{d=1}^{\infty} \frac{1}{d^s \log d} \binom{s+d-1}{d} \\ &\geq \frac{1}{2^{rs}} \frac{q-1}{4} \frac{1}{(s-1)!} \sum_{d=1}^{\infty} \frac{1}{d \log d} \\ &= \infty, \end{aligned}$$

where we used that $\binom{s+d-1}{d} \geq \frac{d^{s-1}}{(s-1)!}$. □

Now we use Lemma 6 and find that the set \overline{M} for our choice of F as in (11) and P as in (12) has measure $\mu_s(\overline{M}) = 1$.

Next we consider the finite collection of s -tuples

$$(\beta_1(k_1, u_1), \dots, \beta_s(k_s, u_s))$$

for $u_1, \dots, u_s = 0, 1, \dots, J$ but not all equal to 0. Note that each of these $\beta_i(k_i, u_i)$ considered as element of $\mathbb{Z}_q[x]$ is 0 or relatively prime to k_i for each (k_1, \dots, k_s) which is an element from P defined above.

Now we use Lemma 4 where we choose \overline{P} as $\overline{P} = P$ from (12) and for any choice of $u_1, \dots, u_s = 0, 1, \dots, J$ but not all equal to 0, we choose the $\beta_i(k_i)$ from Lemma 4 as $\beta_i(k_i) = \beta_i(k_i, u_i)$. Then for the corresponding set $\widetilde{M} := \widetilde{M}(u_1, \dots, u_s)$ of Lemma 4 we have $\mu_s(\widetilde{M}) = 0$.

We set

$$M := \overline{M} \setminus \bigcup_{\substack{u_1, \dots, u_s=0 \\ (u_1, \dots, u_s) \neq (0, \dots, 0)}}^J \widetilde{M}(u_1, \dots, u_s)$$

and find that $\mu_s(M) = 1$.

Now we make a suitable choice for $\mathbf{f} = (f_1, \dots, f_s)$ and for \mathbf{k}^* . Let $(f_1, \dots, f_s) \in M$ and let $(\tilde{k}_1, \dots, \tilde{k}_s) \in P$ be such that

$$\nu(\{\tilde{k}_1 f_1 + \dots + \tilde{k}_s f_s\}) \leq -F(r_1, \dots, r_s) \leq -(r_1 + \dots + r_s)$$

and

$$\nu(\{(\tilde{k}_1 + \beta_1(\tilde{k}_1, u_1))f_1 + \dots + (\tilde{k}_s + \beta_s(\tilde{k}_s, u_s))f_s\}) \geq -\frac{r_1 + \dots + r_s}{2},$$

where $r_i = \deg(\tilde{k}_i)$ and $\tilde{k}_i = q^{\tilde{a}_i-1} + \tilde{\ell}_i$. By the definition of M there are infinitely many such s -tuples $(\tilde{k}_1, \dots, \tilde{k}_s)$.

Let $m := \lfloor F(r_1, \dots, r_s) \rfloor$ and $N = q^{m-1}$. We analyze the first summand in (10): We have

$$-w(\tilde{k}_1, \dots, \tilde{k}_s) = \nu(\{\tilde{k}_1 f_1 + \dots + \tilde{k}_s f_s\}) \leq -F(r_1, \dots, r_s) \leq -m$$

and hence $w \geq m$. This means that $G(N, w(\tilde{k}_1, \dots, \tilde{k}_s)) = N$ and hence we obtain

$$\left| \left[\prod_{j=1}^s \theta(\tilde{k}_j) \right] G(N, w(\tilde{k}_1, \dots, \tilde{k}_s)) \right| \geq c_6(q, s) \frac{N}{q^{\tilde{a}_1 + \dots + \tilde{a}_s}}.$$

Now we turn to the second summand in (10). We have

$$\begin{aligned} & -w(\tilde{k}_1 + \beta_1(\tilde{k}_1, u_1), \dots, \tilde{k}_s + \beta_s(\tilde{k}_s, u_s)) \\ &= \nu(\{(\tilde{k}_1 + \beta_1(\tilde{k}_1, u_1))f_1 + \dots + (\tilde{k}_s + \beta_s(\tilde{k}_s, u_s))f_s\}) \\ &\geq -\frac{r_1 + \dots + r_s}{2} \end{aligned}$$

and hence

$$w(\tilde{k}_1 + \beta_1(\tilde{k}_1, u_1), \dots, \tilde{k}_s + \beta_s(\tilde{k}_s, u_s)) \leq \frac{r_1 + \dots + r_s}{2}.$$

This means that

$$\begin{aligned} |G(N, w(\tilde{k}_1 + \beta_1(\tilde{k}_1, u_1), \dots, \tilde{k}_s + \beta_s(\tilde{k}_s, u_s)))| &\leq c_7(q) q^w \\ &\leq c_7(q) q^{\frac{r_1 + \dots + r_s}{2}} \\ &= c_8(q, s) q^{\frac{\tilde{a}_1 + \dots + \tilde{a}_s}{2}} \end{aligned}$$

and hence we obtain

$$\begin{aligned} & \left| \sum_{\substack{u_1, \dots, u_s \geq 0 \\ \exists j: u_j > J}} \left[\prod_{j=1}^s \theta(\tilde{k}_j + \beta_j(\tilde{k}_j, u_j)) \right] G(N, w(\tilde{k}_1 + \beta_1(\tilde{k}_1, u_1), \dots, \tilde{k}_s + \beta_s(\tilde{k}_s, u_s))) \right| \\ &\leq c_9(q, s) \frac{q^{\frac{\tilde{a}_1 + \dots + \tilde{a}_s}{2}}}{q^{\tilde{a}_1 + \dots + \tilde{a}_s}}. \end{aligned}$$

Altogether we have

$$|\Lambda| \geq c_6(q, s) \frac{N}{q^{\tilde{a}_1 + \dots + \tilde{a}_s}} - c_9(q, s) \frac{q^{\frac{\tilde{a}_1 + \dots + \tilde{a}_s}{2}}}{q^{\tilde{a}_1 + \dots + \tilde{a}_s}} - c_5(q, s) \frac{N}{q^{a_1^* + \dots + a_s^*}} \frac{1}{q^{J/q}}$$

$$\geq c_{10}(q, s) \frac{N}{q^{\tilde{a}_1 + \dots + \tilde{a}_s}}$$

for J large enough and for $\deg(\tilde{k}_1) + \dots + \deg(\tilde{k}_s)$ large enough. Now

$$\begin{aligned} \frac{N}{q^{\tilde{a}_1 + \dots + \tilde{a}_s}} &\geq c_{11}(q, s) q^{F(r_1, \dots, r_s) - r_1 - \dots - r_s} \\ &= c_{11}(q, s) (r_1 + \dots + r_s) \log(r_1 + \dots + r_s) \\ &\geq c_{12}(q, s) (\log N)^s \log \log N. \end{aligned}$$

From the definition of Λ it follows that there exists an $\mathbf{x} \in \mathbb{Q}^s(q^m) \subseteq [0, 1]^s$ such that

$$|D(\mathbf{x}, N)| \geq c_{13}(q, s) (\log N)^s \log \log N$$

and the proof of Theorem 2 is finished. \square

References

- [1] J. Beck: Probabilistic diophantine approximation, I. Kronecker-sequences. *Ann. Math.* 140: 451–502, 1994.
- [2] L. Carlitz: The arithmetic of polynomials in a Galois field. *Amer. J. Math.* 54: 39–50, 1932.
- [3] J. Dick and F. Pillichshammer: *Digital Nets and Sequences. Discrepancy Theory and Quasi-Monte Carlo Integration*. Cambridge University Press, Cambridge, 2010.
- [4] P. Kritzer and F. Pillichshammer: Low discrepancy polynomial lattice point sets. *J. Number Theory* 132: 2510–2534, 2012.
- [5] L. Kuipers and H. Niederreiter: *Uniform Distribution of Sequences*. John Wiley, New York, 1974; reprint, Dover Publications, Mineola, NY, 2006.
- [6] G. Larcher: Nets obtained from rational functions over finite fields. *Acta Arith.* 63: 1–13, 1993.
- [7] G. Larcher: On the distribution of an analog to classical Kronecker-sequences. *J. Number Theory* 52: 198–215, 1995.
- [8] G. Larcher and H. Niederreiter: Kronecker-type sequences and nonarchimedean diophantine approximation. *Acta Arith.* 63: 380–396, 1993.
- [9] G. Larcher and F. Pillichshammer: A metrical best possible lower bound on the star discrepancy of digital sequences. submitted, 2013.
- [10] H. Niederreiter: Point sets and sequences with small discrepancy. *Monatsh. Math.* 104: 273–337, 1987.
- [11] H. Niederreiter: *Random Number Generation and Quasi-Monte Carlo Methods*. SIAM, Philadelphia, 1992.

- [12] K.F. Roth: On irregularities of distribution. *Mathematica* 1: 73–79, 1954.
- [13] W.M. Schmidt: Irregularities of distribution VII. *Acta Arith.* 21: 45–50, 1972.
- [14] V. G. Sprindžuk: *Metric Theory of Diophantine Approximations*. Scripta Series in Mathematics. V. H. Winston & Sons, Washington, D.C.; A Halsted Press Book, John Wiley & Sons, New York-Toronto, Ont.-London, 1979.
- [15] H. Weyl: Über die Gleichverteilung von Zahlen modulo Eins. *Math. Ann.* 77: 313–352, 1916.

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